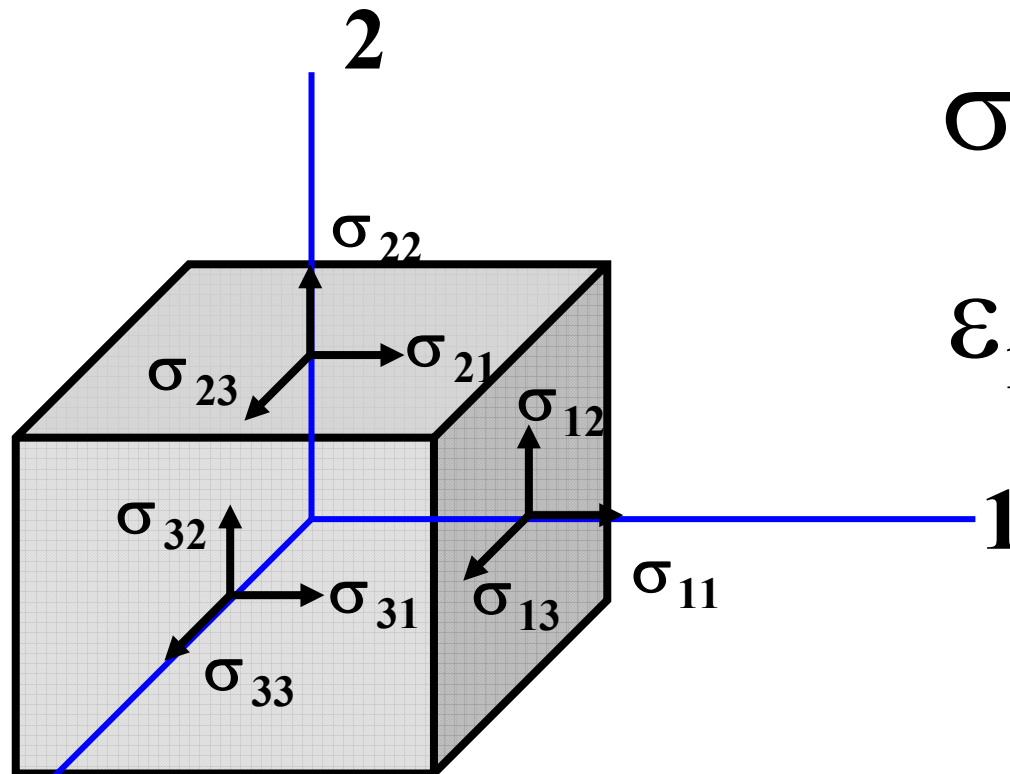


Introduction to stress analysis in composites

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Università di Padova**

Stato di tensione generico



$$\sigma_{ij} \quad i, j = 1, 2, 3$$

$$\varepsilon_{ij} \quad i, j = 1, 2, 3$$

3

$$\sigma_{ij} = f_{ij} (\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}, \varepsilon_{31}, \varepsilon_{32}, \varepsilon_{33})$$



Legge di Hooke generalizzata

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{21} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} & C_{1132} & C_{1113} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} & C_{2232} & C_{2213} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} & C_{3332} & C_{3313} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} & C_{2332} & C_{2313} & C_{2321} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} & C_{3132} & C_{3113} & C_{3121} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} & C_{1232} & C_{1213} & C_{1221} \\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3231} & C_{3212} & C_{3232} & C_{3213} & C_{3221} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1331} & C_{1312} & C_{1332} & C_{1313} & C_{1321} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2131} & C_{2112} & C_{2132} & C_{2113} & C_{2121} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{32} \\ \varepsilon_{13} \\ \varepsilon_{21} \end{Bmatrix}$$



Simmetria e notazione contratta

$$\sigma_{ij} = \sigma_{ji} \quad i \neq j \quad \varepsilon_{ij} = \varepsilon_{ji} \quad i \neq j$$

$$\sigma_{11} = \sigma_1$$

$$\varepsilon_{11} = \varepsilon_1$$

$$\sigma_{22} = \sigma_2$$

$$\varepsilon_{22} = \varepsilon_2$$

$$\sigma_{33} = \sigma_3$$

$$\varepsilon_{33} = \varepsilon_3$$

$$\sigma_{23} = \sigma_{32} = \sigma_4$$

$$2\varepsilon_{23} = 2\varepsilon_{32} = \gamma_{23} = \gamma_{32} = \varepsilon_4$$

$$\sigma_{31} = \sigma_{13} = \sigma_5$$

$$2\varepsilon_{13} = 2\varepsilon_{31} = \gamma_{13} = \gamma_{31} = \varepsilon_5$$

$$\sigma_{12} = \sigma_{21} = \sigma_6$$

$$2\varepsilon_{12} = 2\varepsilon_{21} = \gamma_{12} = \gamma_{21} = \varepsilon_6$$



Legge di Hooke generalizzata

$$\sigma_i = C_{ij}\varepsilon_j \quad i, j = 1, 2, \dots, 6 \quad \text{oppure} \quad \{\sigma\} = [C]\{\varepsilon\}$$

$$\varepsilon_i = S_{ij}\sigma_j \quad i, j = 1, 2, \dots, 6$$

or

$$\{\varepsilon\} = [S]\{\sigma\}$$

$$[S] = [C]^{-1}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$



Simmetria matrice di rigidezza

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ \vdots & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ \vdots & \dots & C_{33} & C_{34} & C_{35} & C_{36} \\ \vdots & \dots & \dots & C_{44} & C_{45} & C_{46} \\ \vdots & \mathbf{SYM} & \dots & \dots & C_{55} & C_{56} \\ \vdots & \dots & \dots & \dots & \dots & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

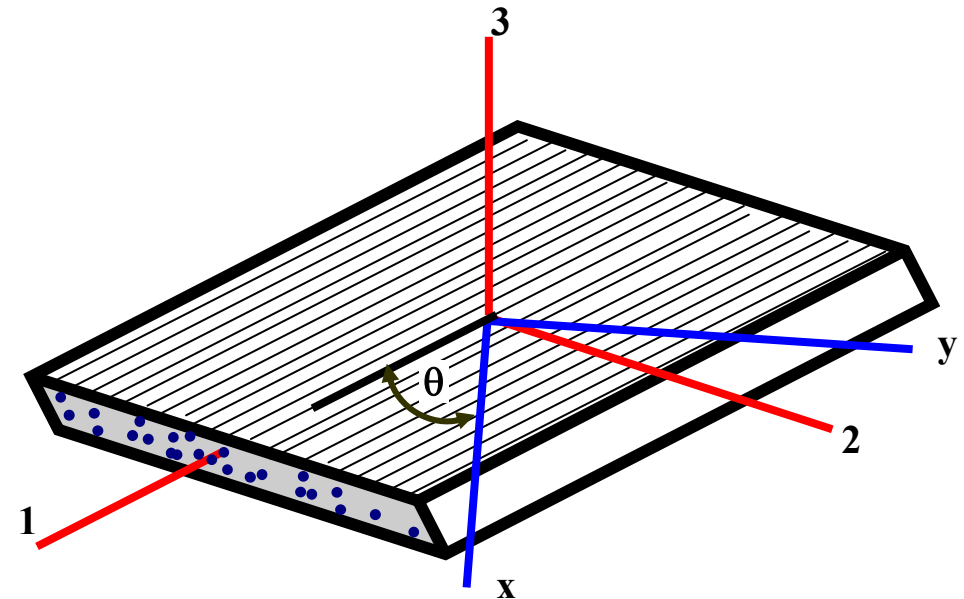
21 costanti indipendenti



Materiale ortotropo

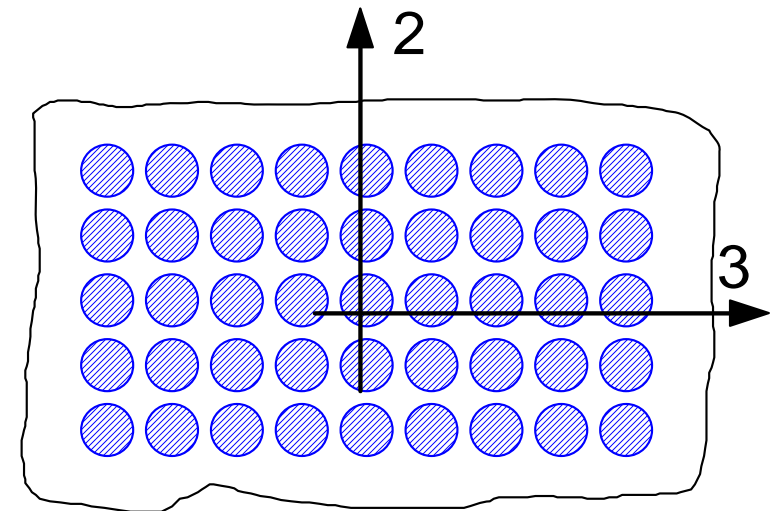
Tre piani di simmetria
9 costanti indipendenti

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ \vdots & C_{22} & C_{23} & 0 & 0 & 0 \\ \vdots & \dots & C_{33} & 0 & 0 & 0 \\ \vdots & \dots & \dots & C_{44} & 0 & 0 \\ \vdots & \text{SYM} & \dots & \dots & C_{55} & 0 \\ \vdots & \dots & \dots & \dots & \dots & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$



Materiale trasversalmente isotropo

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ \vdots & C_{22} & C_{23} & 0 & 0 & 0 \\ \vdots & \dots & C_{22} & 0 & 0 & 0 \\ \vdots & \dots & \dots & \frac{(C_{22}-C_{23})}{2} & 0 & 0 \\ \vdots & \text{SYM} & \dots & \dots & C_{66} & 0 \\ \vdots & \dots & \dots & \dots & \dots & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$



Le direzioni 2 e 3 coincidono
5 costanti indipendenti



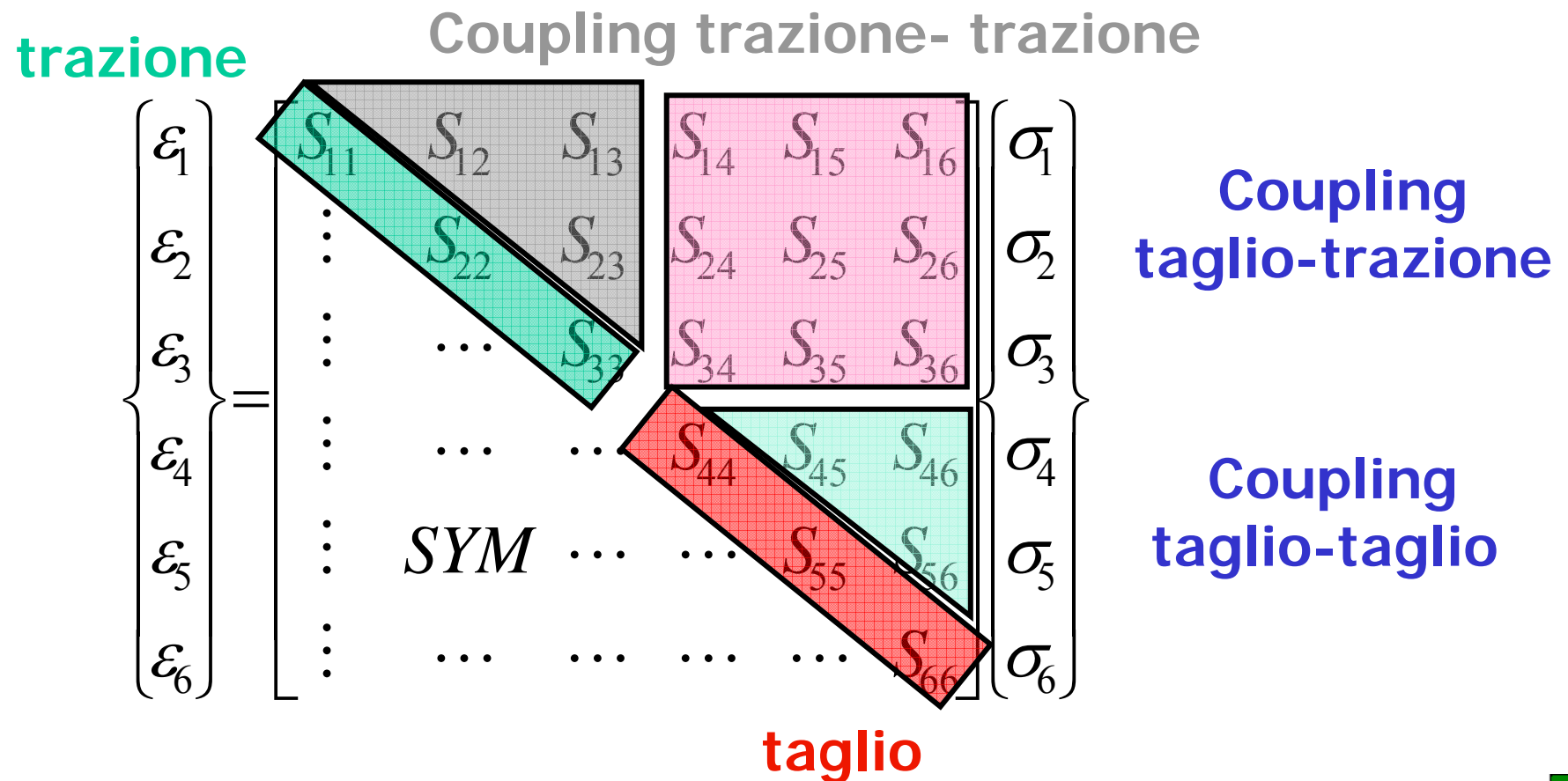
Materiale isotropo

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ \vdots & C_{11} & C_{12} & 0 & 0 & 0 \\ \vdots & \dots & C_{11} & 0 & 0 & 0 \\ \vdots & \dots & \dots & \frac{(C_{11}-C_{12})}{2} & 0 & 0 \\ \vdots & \text{SYM} & \dots & \dots & \frac{(C_{11}-C_{12})}{2} & 0 \\ \vdots & \dots & \dots & \dots & \dots & \frac{(C_{11}-C_{12})}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

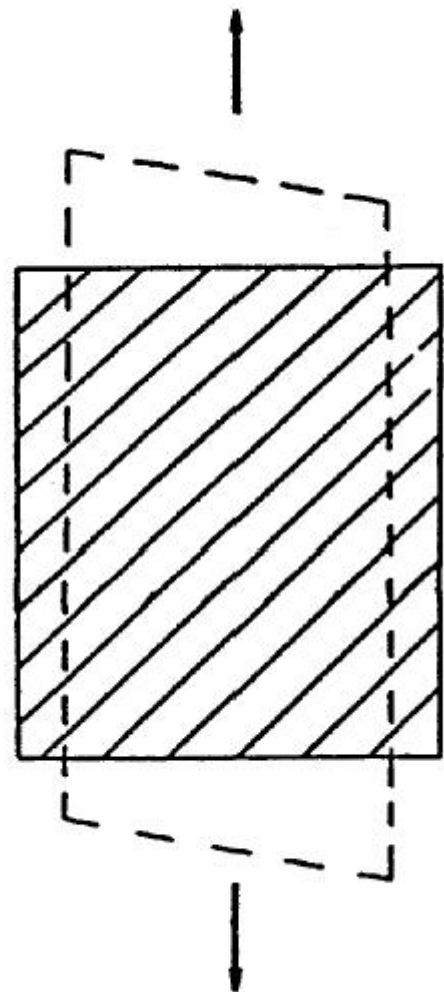
2 costanti indipendenti



Materiale anisotropo



Coupling trazione-taglio



Deformata per effetto di una sollecitazione di trazione

Materiale ortotropo matrice in termini di costanti ingegneristiche



$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \quad \frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}$$



Materiale trasversalmente isotropo

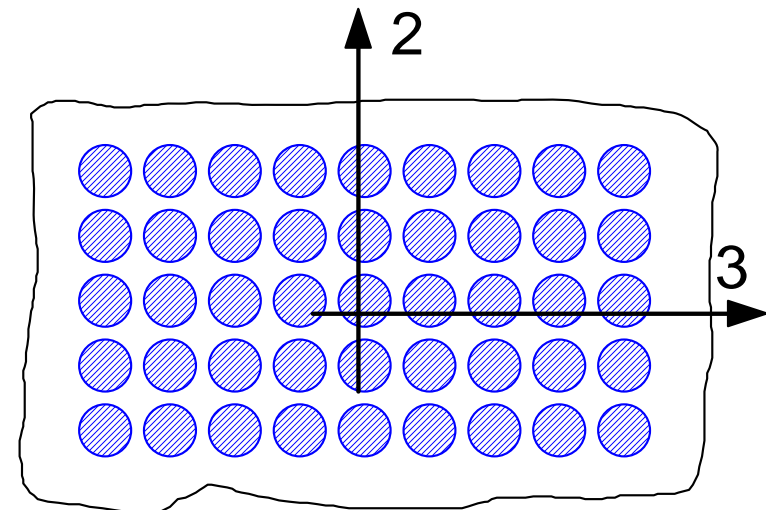
$$\mathbf{E}_2 = \mathbf{E}_3$$

$$\mathbf{G}_{12} = \mathbf{G}_{13}$$

$$\nu_{21} = \nu_{31}$$

$$\nu_{23} = \nu_{32}$$

$$\mathbf{G}_{23} = \frac{\mathbf{E}_2}{2(1 + \nu_{32})}$$



Materiale ortotropo in stato piano di tensione

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} & \mathbf{0} \\ \mathbf{S}_{21} & \mathbf{S}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix}$$

$$S_{11} = \frac{1}{E_1}$$

$$S_{22} = \frac{1}{E_2}$$

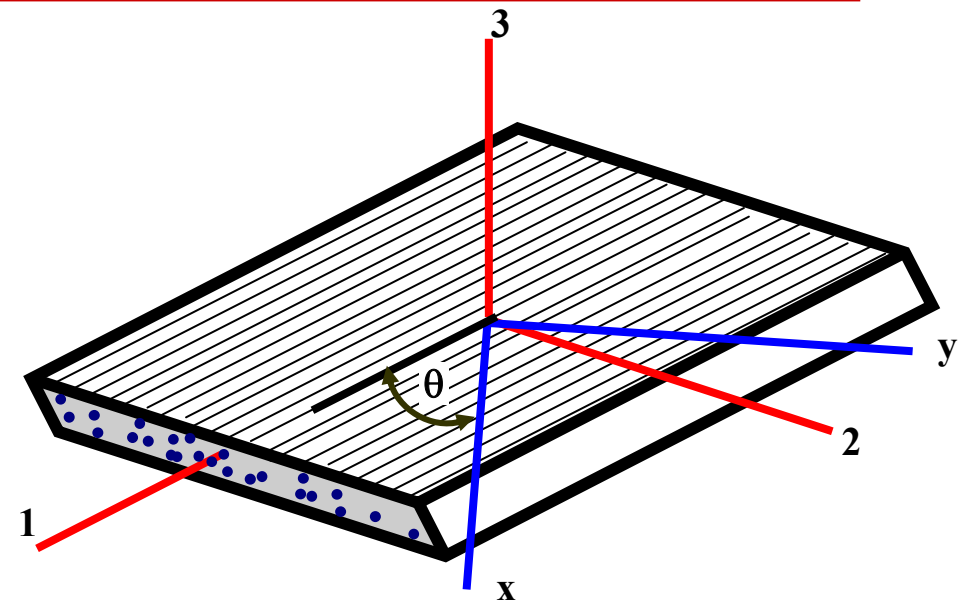
4 costanti indipendenti

$$S_{12} = S_{21} = -\frac{\nu_{21}}{E_2} = -\frac{\nu_{12}}{E_1}$$

$$S_{66} = \frac{1}{G_{12}}$$



Una lamina di composito unidirezionale o tessuto può essere considerata come materiale ortotropo in stato piano di tensione!



$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$



Termini della matrice di rigidezza

$$Q_{11} = \frac{S_{22}}{(S_{11}S_{22} - S_{12}^2)} = \frac{E_1}{(1 - \nu_{12}\nu_{21})}$$

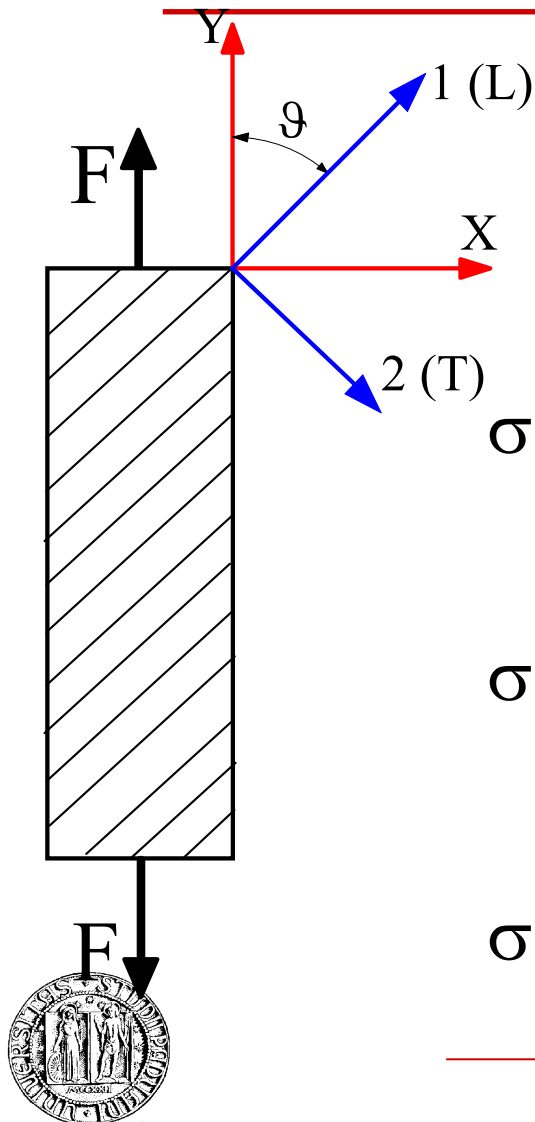
$$Q_{22} = \frac{S_{11}}{(S_{11}S_{22} - S_{12}^2)} = \frac{E_2}{(1 - \nu_{12}\nu_{21})}$$

$$Q_{12} = Q_{21} = -\frac{S_{12}}{(S_{11}S_{22} - S_{12}^2)} = \frac{\nu_{12}E_2}{(1 - \nu_{12}\nu_{21})}$$

$$Q_{66} = \frac{1}{S_{66}} = G_{12}$$



Rotazione sistema di riferimento



$$\sigma_x = \sigma_1 \cos^2 \theta + \sigma_2 \sin^2 \theta - 2\sigma_{12} \sin \theta \cos \theta$$

$$\sigma_{xy} = \sigma_1 \sin \theta \cos \theta - \sigma_2 \sin \theta \cos \theta + \sigma_{12} (\cos^2 \theta - \sin^2 \theta)$$

$$\sigma_y = \sigma_1 \sin^2 \theta + \sigma_2 \cos^2 \theta + 2\sigma_{12} \cos \theta \sin \theta$$

Matrice di trasformazione

$$[T] = \begin{bmatrix} \cos^2 \vartheta & \sin^2 \vartheta & 2\sin \vartheta \cos \vartheta \\ \sin^2 \vartheta & \cos^2 \vartheta & -2\sin \vartheta \cos \vartheta \\ -\sin \vartheta \cos \vartheta & \sin \vartheta \cos \vartheta & \cos^2 \vartheta - \sin^2 \vartheta \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} \quad \text{oppure} \quad \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix}$$



Deformazioni tensoriali ed ingegneristiche

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix}$$

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [R] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [R] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix}$$



Relazione tensioni-deformazioni – I

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{Q}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$[\bar{Q}] = [T]^{-1} [Q] [R] [T] [R]^{-1}$$

**matrice di rigidezza della lamina
nel sistema di riferimento di struttura**



Relazione tensioni-deformazioni - II

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

con, ad esempio:

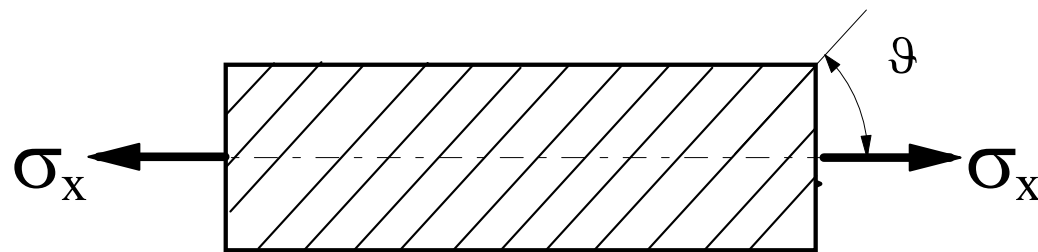
$$\bar{Q}_{11} = Q_{11} \cos^4 \theta + Q_{22} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta$$



$$\begin{aligned} \bar{Q}_{11} &= Q_{11} \cos^4 \theta + Q_{22} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta \\ \bar{Q}_{22} &= Q_{11} \sin^4 \theta + Q_{22} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12}(\cos^4 \theta + \sin^4 \theta) \\ \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66}(\sin^4 \theta + \cos^4 \theta) \\ \bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) \cos^3 \theta \sin \theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos \theta \sin^3 \theta \\ \bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) \cos \theta \sin^3 \theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos^3 \theta \sin \theta \end{aligned} \quad (1)$$



Proprietà apparenti della lamina



$$\sigma_1 = \sigma_x \cos^2 \theta$$

$$\sigma_2 = \sigma_x \sin^2 \theta$$

$$\tau_{12} = -\sigma_x \sin \theta \cos \theta$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} = f(E_L, E_T, \nu_{LT}, G_{LT}, \sigma_x, \theta)$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix} = g(E_L, E_T, \nu_{LT}, G_{LT}, \sigma_x, \theta)$$

$$E_x(\vartheta) = \frac{\sigma_x}{\varepsilon_x} = \left[\frac{\cos^4 \vartheta}{E_L} + \frac{\sin^4 \vartheta}{E_T} + \frac{1}{4} \left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L} \right) \sin^2 2\vartheta \right]^{-1}$$

$$\frac{\nu_{xy}(\vartheta)}{E_x(\vartheta)} = \frac{\nu_{LT}}{E_L} - \frac{1}{4} \left(\frac{1}{E_L} + \frac{2\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}} \right) \sin^2 2\vartheta$$



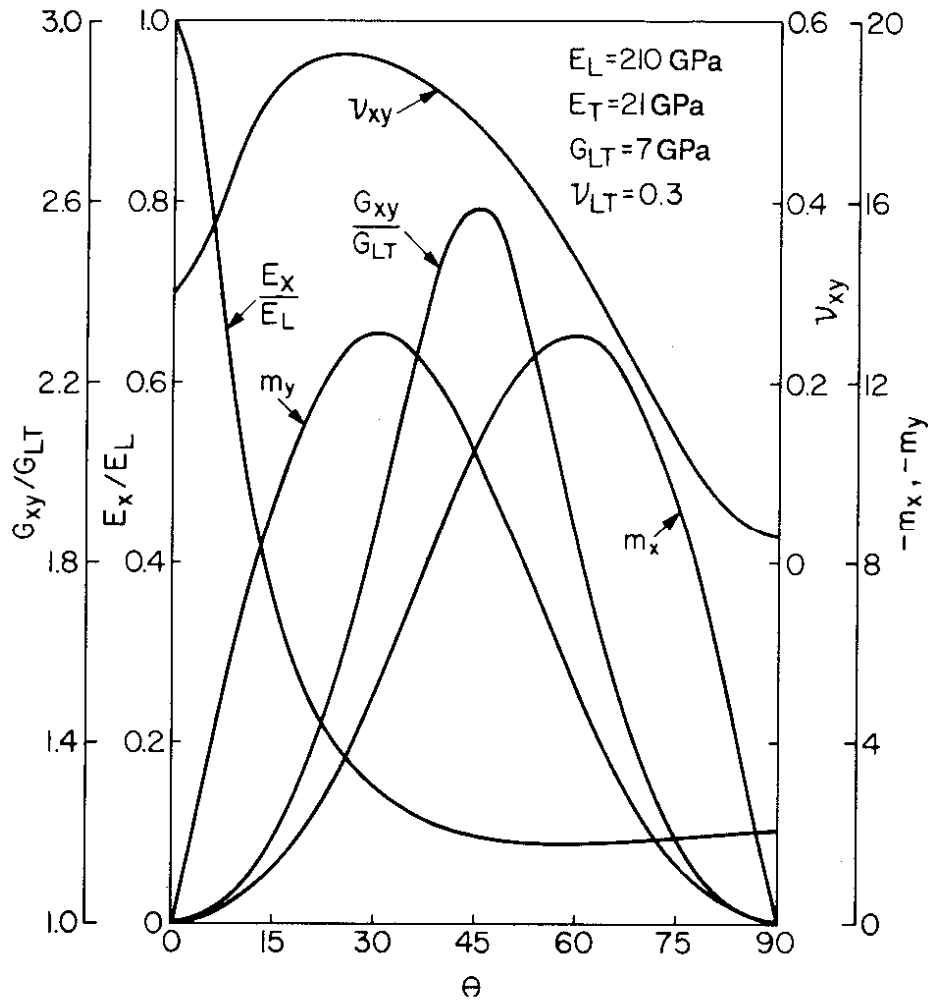


Figure 5.11. Variation in elastic constants of boron-epoxy system

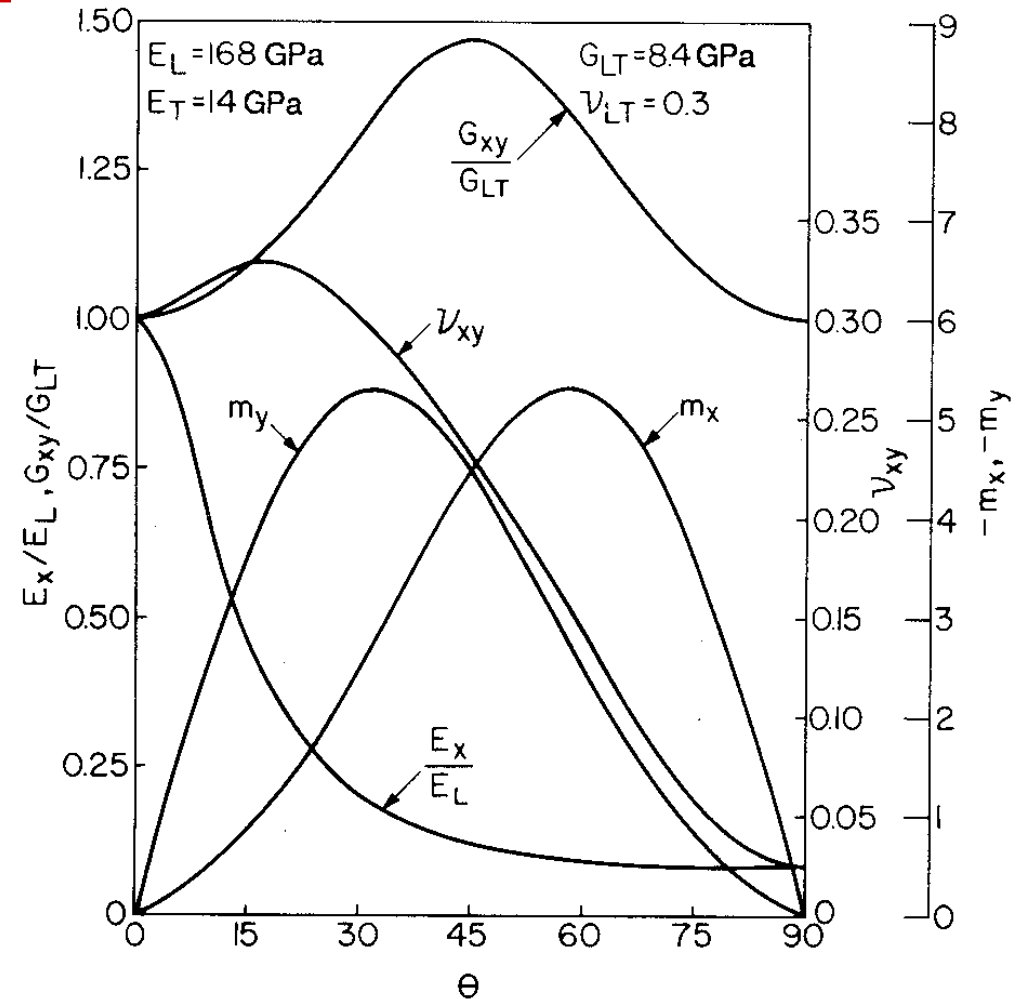


Figure 5.10. Variation in elastic constants of graphite-epoxy system

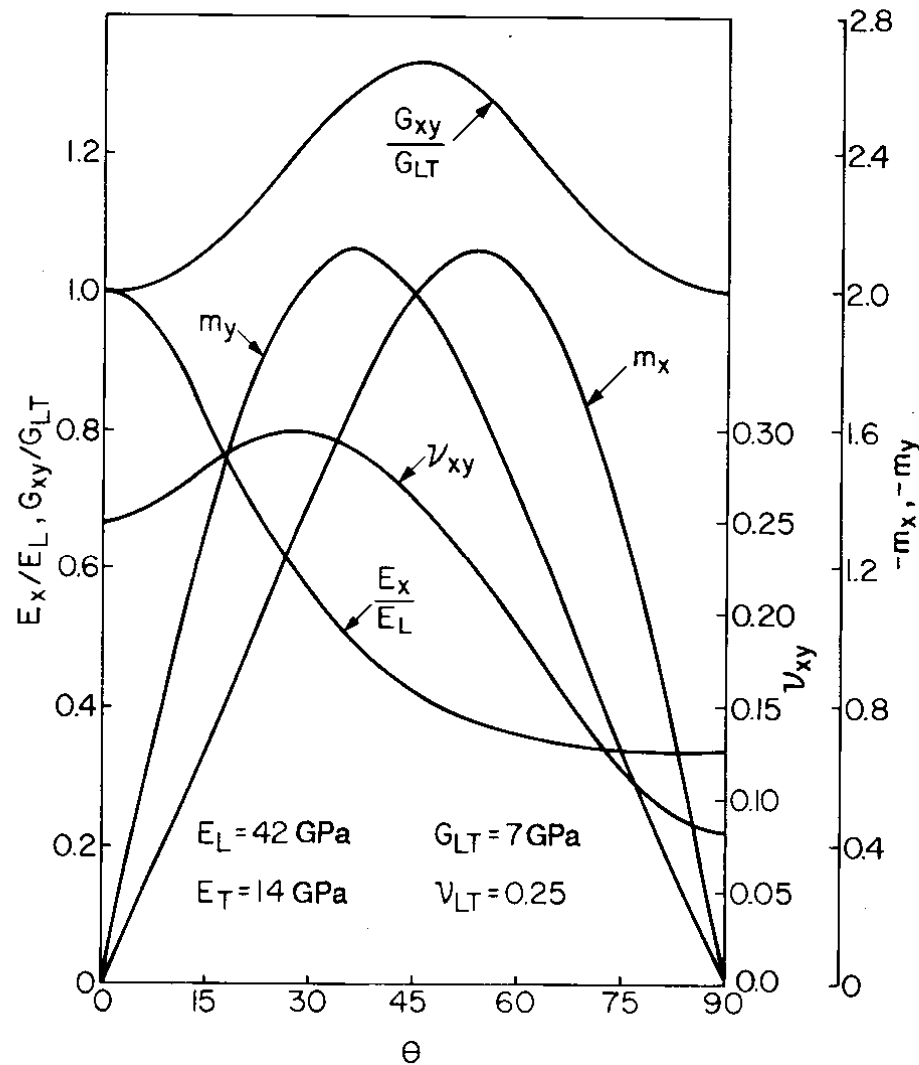


Figure 5.9. Variation in elastic constants of glass-epoxy systems.

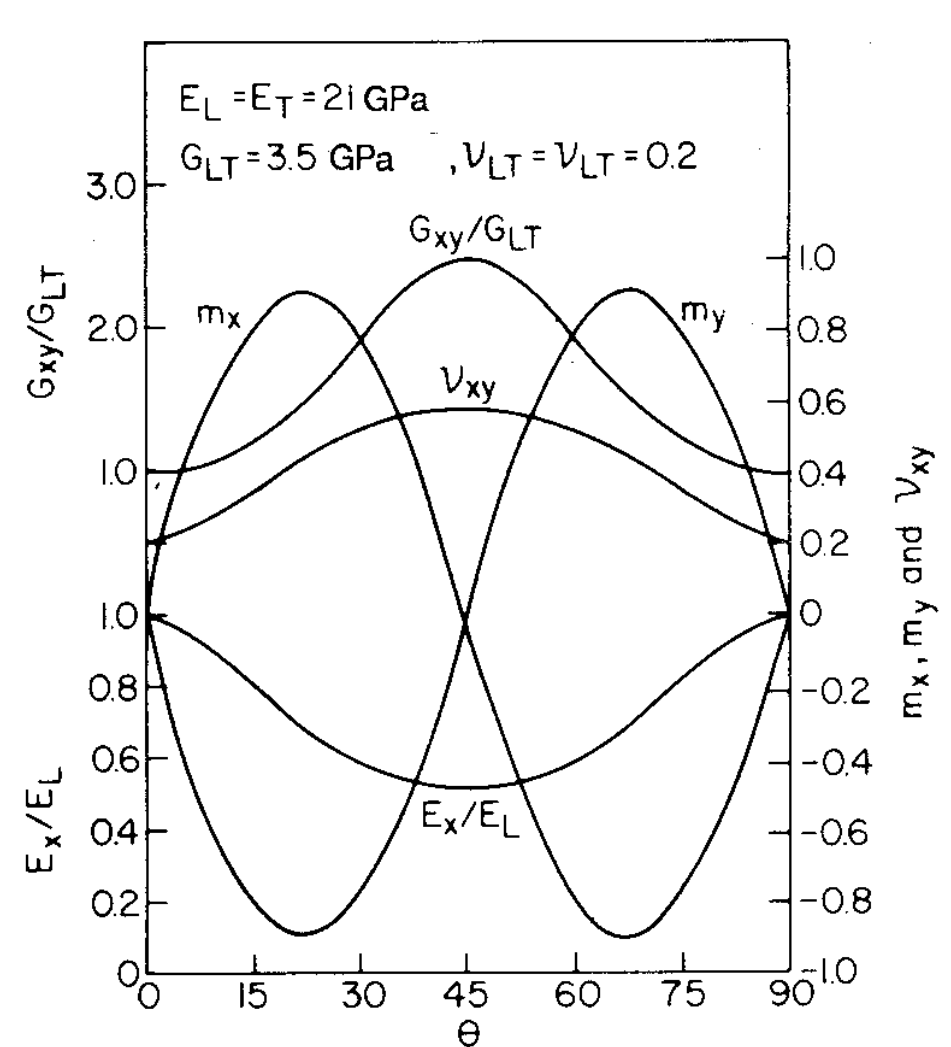
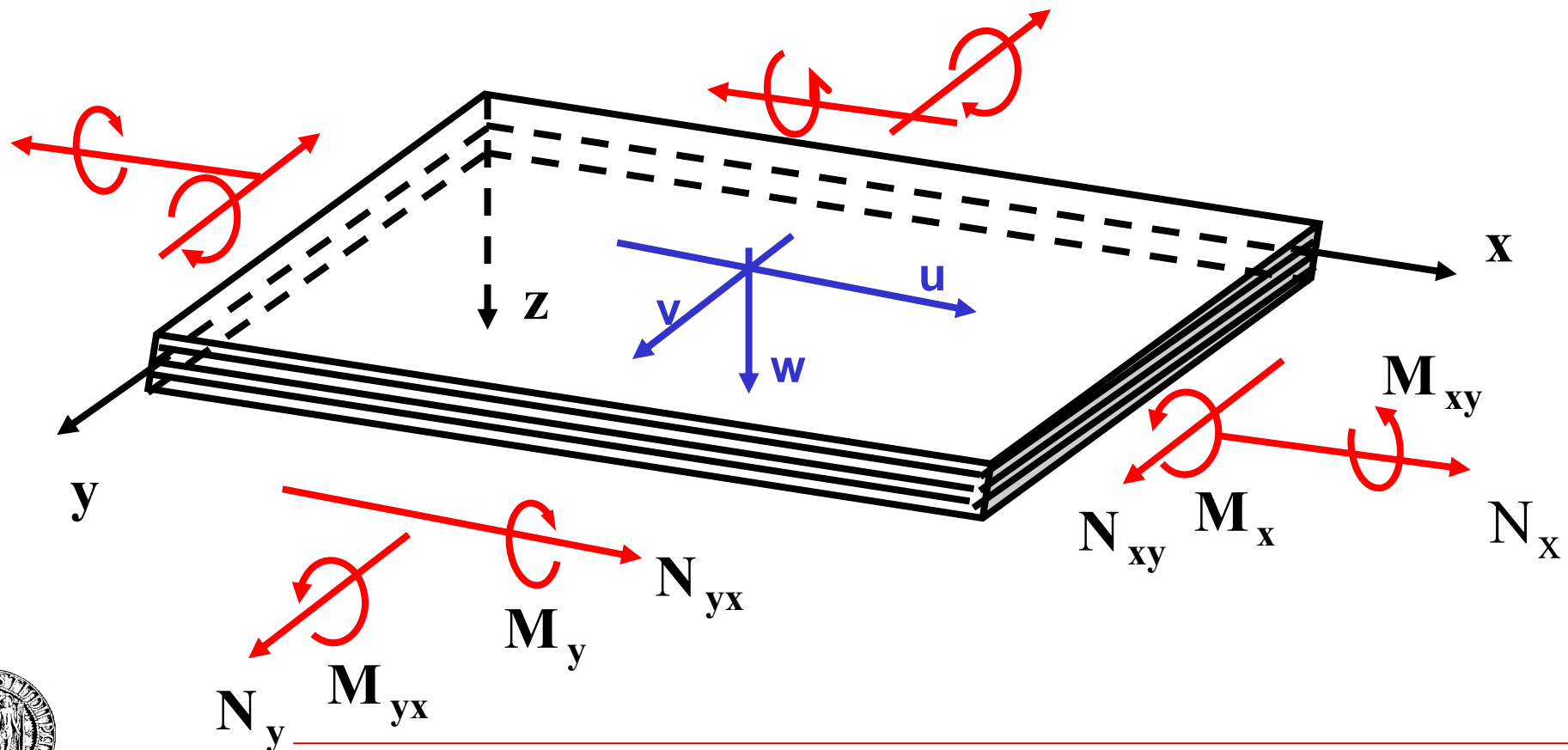


Figure 5.12. Variation in elastic constants of balanced lamina.

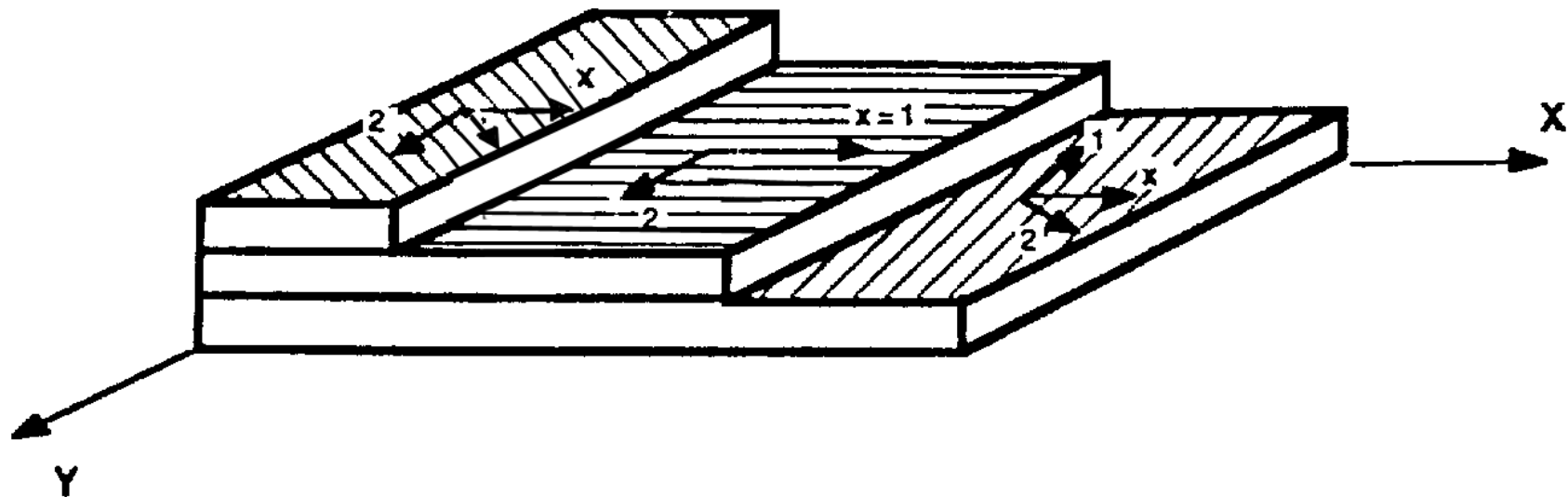
Analisi di un laminato

Sistema di coordinate e sollecitazioni applicate ad un laminato



Laminato multistrato

L'orientazione e le caratteristiche elastiche di ciascuno strato possono essere variabili!



Teoria della laminazione e ipotesi semplificative



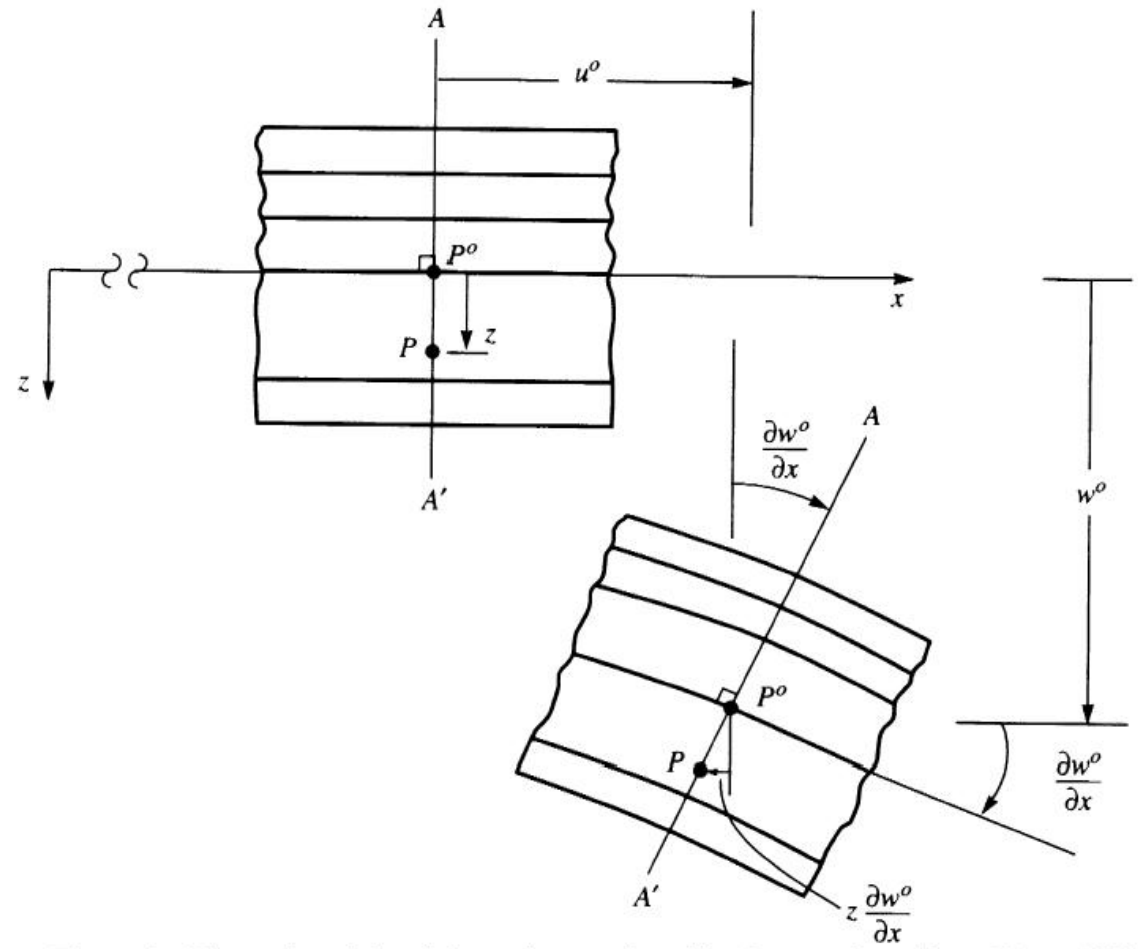
- materiale omogeneo ed ortotropo per ogni lamina;
- comportamento elastico lineare;
- spessore del laminato piccolo rispetto alle altre dimensioni e costante;
- spostamenti piccoli rispetto allo spessore del laminato;
- stato di tensione piano per ciascuna lamina;
- deformazione ε_z trascurabile;
- spostamenti in direzione x e y funzioni lineari dello spostamento lungo l'asse z ; (sezioni piane restano piane)
- scorrimenti γ_{xz} , γ_{yz} trascurabili;
- taglio interlaminare trascurabile



Componenti di spostamento

$$u = u^0 - z \frac{\partial w}{\partial x}$$

$$v = v^0 - z \frac{\partial w}{\partial y}$$



Deformazioni nel laminato

$$\varepsilon_x = \frac{\partial u}{\partial x} = \varepsilon_x^0 + z \cdot k_x$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \varepsilon_y^0 + z \cdot k_y$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy}^0 + z \cdot k_{xy}$$



Componenti di deformazione

Deformazioni nel piano medio

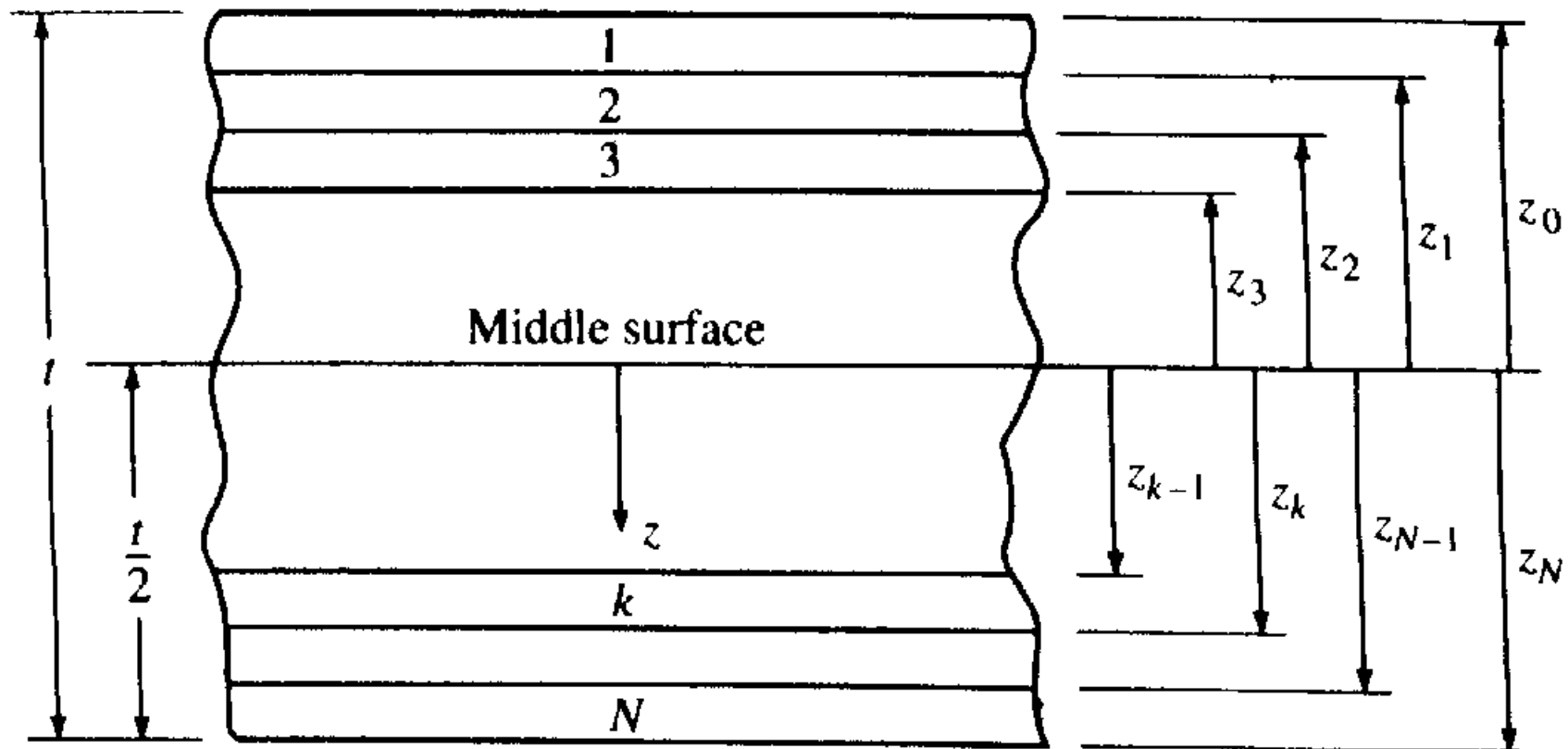
$$\varepsilon_x^0 = \frac{\partial u^0}{\partial x}$$
$$\varepsilon_y^0 = \frac{\partial v^0}{\partial y}$$
$$\gamma_{xy}^0 = \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}$$

Curvature

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}$$
$$\kappa_y = -\frac{\partial^2 w}{\partial y^2}$$
$$\kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$



Geometria del laminato e numerazione degli strati



Stato di tensione nella lamina *i*-esima

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 + z\kappa_x \\ \varepsilon_y^0 + z\kappa_y \\ \gamma_{xy}^0 + z\kappa_{xy} \end{Bmatrix}$$



Sollecitazioni risultanti

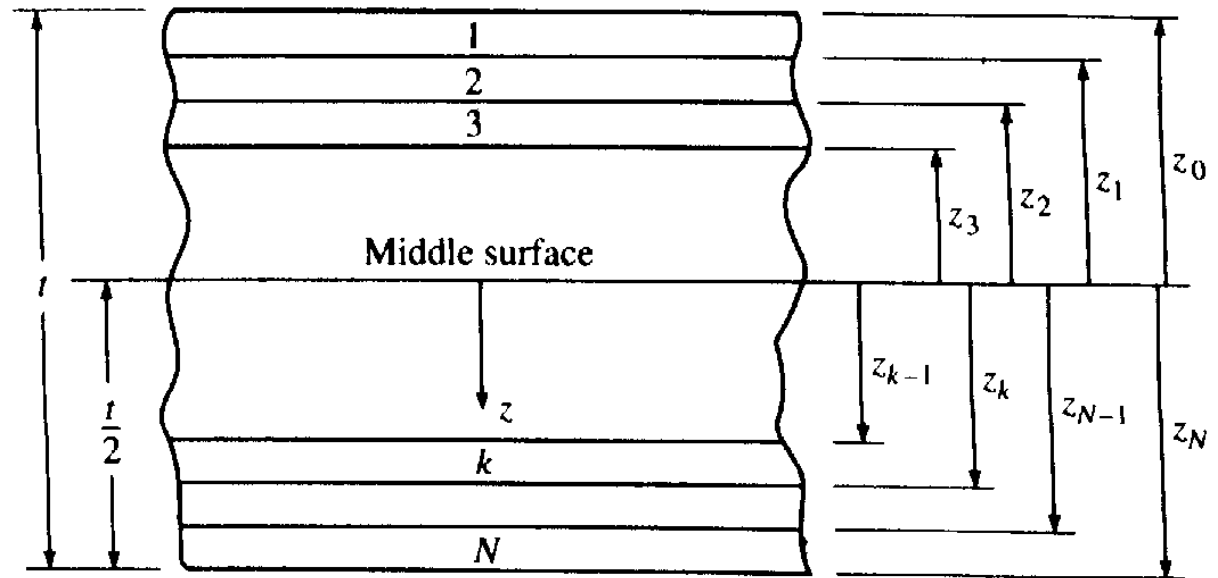
Forza per unità di lunghezza :

$$N_x = \int_{-t/2}^{t/2} \sigma_x dz = \sum_{k=1}^N \left\{ \int_{z_{k-1}}^{z_k} (\sigma_x)_k dz \right\}$$

Momento per unità di lunghezza :

$$M_x = \int_{-t/2}^{t/2} \sigma_x z dz = \sum_{k=1}^N \left\{ \int_{z_{k-1}}^{z_k} (\sigma_x)_k z dz \right\}$$





dove :

t = spessore laminato

$(\sigma_x)_k$ = stress nella k esima lamina

z_{k-1} = distanza dal centro del laminato alla superficie interna del k esima lamina

z_k = distanza dal centro del laminato alla superficie esterna del k esima lamina



- Si ottiene, ad esempio:

$$\mathbf{N}_x = \sum_{k=1}^N \left\{ \int_{z_{k-1}}^{z_k} \left\{ (\bar{\mathbf{Q}}_{11})_k (\varepsilon_x^0 + z\kappa_x) + (\bar{\mathbf{Q}}_{12})_k (\varepsilon_y^0 + z\kappa_y) + (\bar{\mathbf{Q}}_{16})_k (\gamma_{xy}^0 + z\kappa_{xy}) \right\} dz \right\}$$

$$\mathbf{M}_x = \sum_{k=1}^N \left\{ \int_{z_{k-1}}^{z_k} \left\{ (\bar{\mathbf{Q}}_{11})_k (\varepsilon_x^0 + z\kappa_x) + (\bar{\mathbf{Q}}_{12})_k (\varepsilon_y^0 + z\kappa_y) + (\bar{\mathbf{Q}}_{16})_k (\gamma_{xy}^0 + z\kappa_{xy}) \right\} z dz \right\}$$

$$\mathbf{N}_x = \mathbf{A}_{11}\varepsilon_x^0 + \mathbf{A}_{12}\varepsilon_y^0 + \mathbf{A}_{16}\gamma_{xy}^0 + \mathbf{B}_{11}\kappa_x + \mathbf{B}_{12}\kappa_y + \mathbf{B}_{16}\kappa_{xy}$$

$$\mathbf{M}_x = \mathbf{B}_{11}\varepsilon_x^0 + \mathbf{B}_{12}\varepsilon_y^0 + \mathbf{B}_{16}\gamma_{xy}^0 + \mathbf{D}_{11}\kappa_x + \mathbf{D}_{12}\kappa_y + \mathbf{D}_{16}\kappa_{xy}$$

e per gli altri parametri di sollecitazione si procede in maniera analoga



Relazione completa sollecitazioni- deformazioni

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$



Elementi della matrice

$$\mathbf{A}_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} (\overline{\mathbf{Q}}_{ij})_k \, dz = \sum_{k=1}^N (\overline{\mathbf{Q}}_{ij})_k (z_k - z_{k-1}) \quad \text{trazione}$$

$$\mathbf{D}_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} (\overline{\mathbf{Q}}_{ij})_k z^2 \, dz = \frac{1}{3} \left[\sum_{k=1}^N (\overline{\mathbf{Q}}_{ij})_k (z_k^3 - z_{k-1}^3) \right] \quad \text{flessione}$$

$$\mathbf{B}_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} (\overline{\mathbf{Q}}_{ij})_k z \, dz = \frac{1}{2} \left[\sum_{k=1}^N (\overline{\mathbf{Q}}_{ij})_k (z_k^2 - z_{k-1}^2) \right] \quad \begin{array}{l} \text{Coupling} \\ \text{trazione-flessione} \end{array}$$



Forma sintetica

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} = [\mathbf{E}] \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix}$$



Laminati simmetrici - I

- Un laminato si definisce simmetrico se le proprietà geometriche ed elastiche sono simmetriche rispetto al piano medio
- In questo caso non c'è coupling trazione-flessione e gli elementi della matrice B sono nulli: $B_{ij}=0$

$$[B] = 0$$

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix}$$

$$\{N\} = [A]\{\varepsilon^0\}$$

$$\{M\} = [D]\{\kappa\}$$

[45/-45/-45/45]

[0/-45/90/-45/0]

[90/0/60/-60/0/0/-60/60/0/90]



Laminati simmetrici - II

$$\mathbf{B}_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} (\bar{\mathbf{Q}}_{ij})_k \mathbf{z} dz = \frac{1}{2} \left[\sum_{k=1}^N (\bar{\mathbf{Q}}_{ij})_k (z_k^2 - z_{k-1}^2) \right]$$

Lamine simmetricamente disposte rispetto al piano medio con uguale spessore e uguale $[\bar{\mathbf{Q}}]$ (*stesso angolo di inclinazione*) danno contributi opposti a $[\mathbf{B}]$ poiché il termine $(z_k^2 - z_{k-1}^2)$ cambia segno



Laminati simmetrici - III

L'accoppiamento trazione-flessione è dovuto più ad una eventuale non-simmetria di disposizione rispetto al piano medio che alle proprietà ortotrope della lamina ed è quindi presente anche nel caso di laminati in materiale isotropo non simmetrici (es. lamine bimetalliche)



Laminati “Ortotropi” - I

Assenza di accoppiamento trazione-taglio

È necessario che A_{16} e A_{26} siano uguali a zero

$$A_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} (\bar{Q}_{ij})_k dz = \sum_{k=1}^N (\bar{Q}_{ij})_k (z_k - z_{k-1})$$



$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$



$$\begin{aligned} \bar{Q}_{11} &= Q_{11} \cos^4 \theta + Q_{22} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta \\ \bar{Q}_{22} &= Q_{11} \sin^4 \theta + Q_{22} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12}(\cos^4 \theta + \sin^4 \theta) \\ \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66}(\sin^4 \theta + \cos^4 \theta) \\ \bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) \cos^3 \theta \sin \theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos \theta \sin^3 \theta \\ \bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) \cos \theta \sin^3 \theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos^3 \theta \sin \theta \end{aligned} \quad (1)$$

$\bar{Q}_{11}, \bar{Q}_{22}, \bar{Q}_{12}$ e \bar{Q}_{66} Sempre positivi e maggiori di zero ($E_L \gg G_{LT}$)

\bar{Q}_{16} e \bar{Q}_{26}

Funzioni dispari di θ $\bar{Q}_{16}(\theta) = -\bar{Q}_{16}(-\theta)$
e sono nulli per $\theta = 0^\circ$ e 90°

$(z_k - z_{k-1})$

Non cambia mai segno



Laminati “Ortotropi” - III

A_{16} e A_{26} possono essere resi nulli se, per ogni lamina orientata a $+\theta$ è presente nel laminato una lamina di uguale spessore e proprietà (uguale $[\bar{Q}]$) orientata a $-\theta$.
Non è importante la posizione

Questi laminati possono anche essere simmetrici ottenendo quindi simultaneamente ortotropia e assenza di accoppiamento trazione-flessione



Laminati “Ortotropi” - IV

Unidirezionali $[0]_N$

Cross-ply $[0/90]_N$

Angle-ply $[+ \theta/-\theta]_N$

ortotropia

Ortotropi e simmetrici

Cross-ply $[0/90]_{4S}$

Angle-ply $[+ \theta/-\theta]_{3S}$

ortotropia e assenza di
accoppiamento trazione-flessione



Forma sintetica

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} = [\mathbf{E}] \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix}$$



Determinazione stato di deformazione del laminato

$$\begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M \end{Bmatrix} = [E]^{-1} \begin{Bmatrix} N \\ M \end{Bmatrix}$$

invertendo la relazione appena ottenuta è possibile determinare lo stato di deformazione in funzione delle sollecitazioni applicate (per unità di lunghezza)



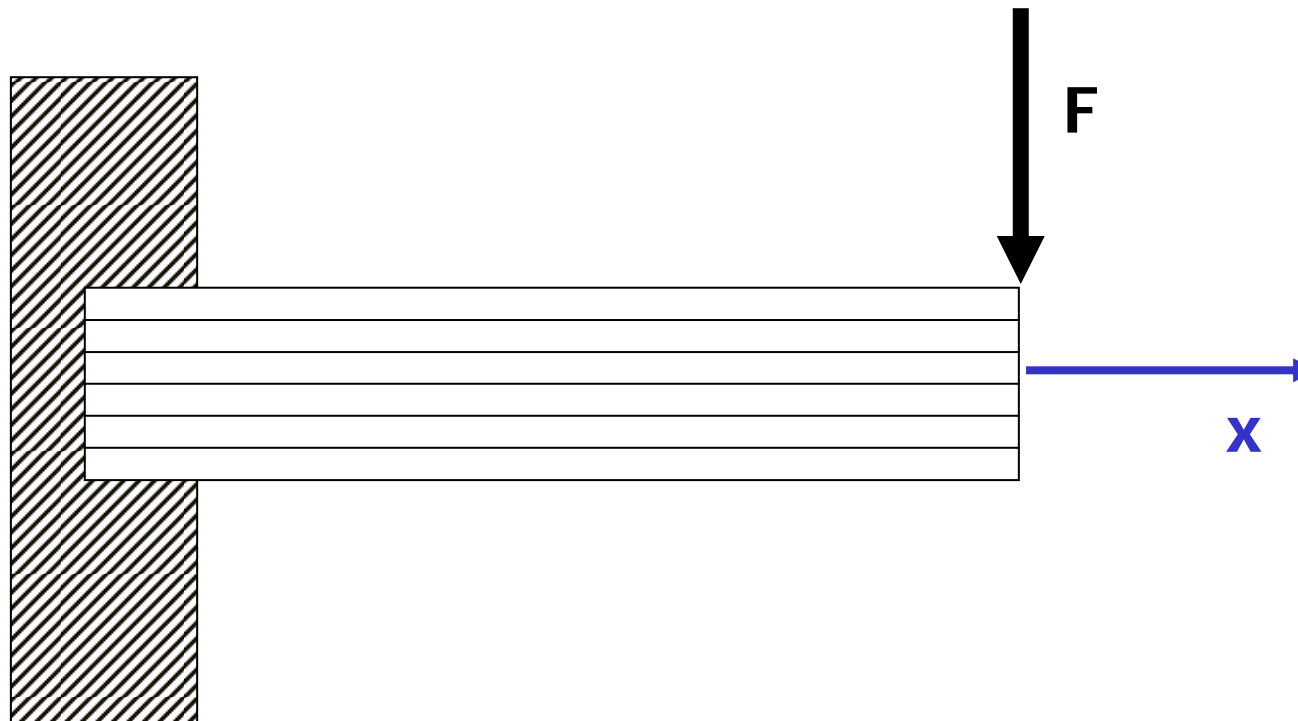
Stato di tensione delle singole lamine

A questo punto è possibile, attraverso le matrici di rigidità di lamina, determinare lo stato di tensione per ciascuna lamina

$$\{\sigma\}_k = [\bar{Q}]_k \left(\{\varepsilon^0\} + z \{\kappa\} \right)$$



Un esempio

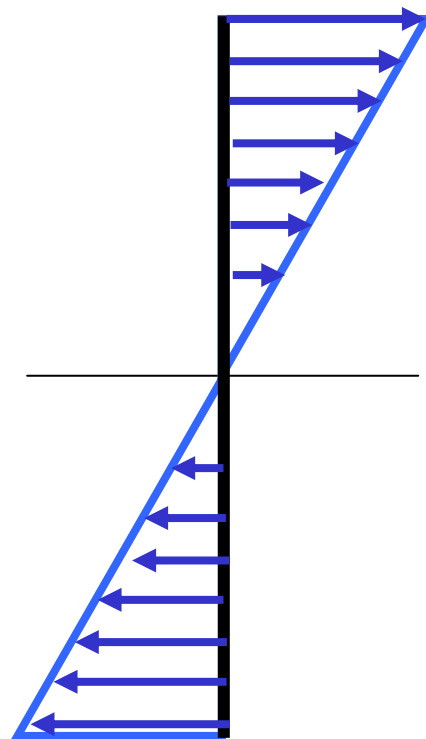


trave a sbalzo a struttura stratificata

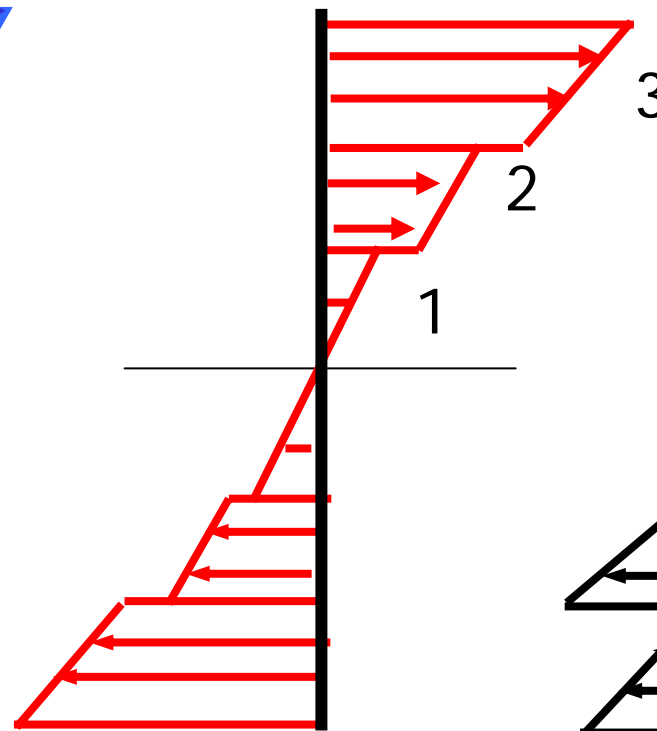


Distribuzioni di tensione

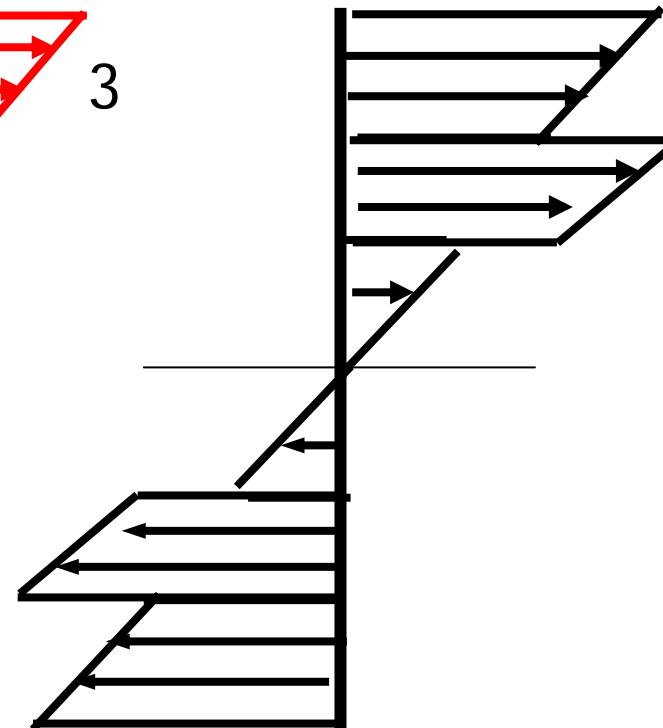
Materiali isotropo



$$(\mathbf{E}_x)_3 > (\mathbf{E}_x)_2 > (\mathbf{E}_x)_1$$



$$(\mathbf{E}_x)_1 = (\mathbf{E}_x)_3 < (\mathbf{E}_x)_2$$



Procedura per l'analisi dello stato di tensione

